Lecture Notes, January 22, 2015

Fundamental Theorems of Welfare Economics

Pareto Efficiency

Definition: An allocation $x^i,\ i \in H$, is attainable if there is $y^j \in Y^j,\ j \in F$ so that $0 \le \sum_{i\in H} x^i \le \sum_{j\in F} y^j + \sum_{i \in H} r^i$. (The inequalities hold co-ordinatewise.)

Definition: Consider two assignments of bundles to consumers, v^i, $w^i \in X^i$, $i \in H$. v^i is said to be Pareto superior to w^i if for each $i \in H$, $u^i(v^i) \ge u^i(w^i)$ and for some $h \in H$, $u^h(v^h) > u^h(w^h)$.

Note that Pareto preferability is an incomplete ordering. There are many allocation pairs that are Pareto incomparable.

Definition: An attainable assignment of bundles to consumers, $w^i,\ i \in H$, is said to be Pareto efficient (or Pareto optimal) if there is no other attainable assignment v^i so that v^i is Pareto superior to w^i.

Definition: $< p^0, x^{0i}, y^{0j} >,\ p^0 \in R^N_+,\ i \in H,\ j \in F$, $x^{0i} \in R^N$, $y^{0j} \in R^N$, is said to be a competitive equilibrium in a private ownership economy if

(i) $y^{0j} \in Y^j$ and $p^0 \cdot y^{oj} \ge p^0 \cdot y$ for all $y \in Y^j$, for all $j \in F$

(ii) $x^{0i} \in X^i,\ M^i(p^0) = p^0 \cdot r^i + \sum_{j \in F} \alpha^{ij}\, p^0 \cdot y^{0j}$

$p^0 \cdot x^{0i} \le M^i(p^0)$

and $u^i(x^{0i}) \ge u^i(x)$ for all $x \in X^i$ with $p^0 \cdot x \le M^i(p^0)$ for all $i \in H$, and

(iii) $0 \ge \sum_{i\in H} x^{0i} - \sum_{j \in F} y^{0j} - \sum_{i \in H} r^i$

(co-ordinatewise) with $p^0_k = 0$ for co-ordinates k so that the strict inequality holds.

This definition is sufficiently general to include the equilibrium developed in each of Theorems 14.1, 18.1, and 24.7. Properties (i) and (ii) embody decentralization. Property (iii) is market clearing.

First Fundamental Theorem of Welfare Economics (1FTWE)

Every competitive equilibrium is Pareto efficient (CE $\Rightarrow$ PE). This result does not require convexity of tastes or technology (though attaining a CE may need convexity).

Theorem 19.1 (First Fundamental Theorem of Welfare Economics): Assume C.II, C.IV, CVI(C) [or subsitute weak monotonicity C.IV* for C.IV and C.VI(C)]. Let $p^0 \in R_+^N$ be a competitive equilibrium price vector of the economy. Let w^{0i}, $i \in H$, be the associated individual consumption bundles, y^{oj}, $j \in F$, be the associated firm supply vectors. Then w^{0i} is Pareto efficient.

Intuition for the proof: Proof by contradiction. If there's a better attainable consumption plan it must be more expensive than CE consumption plan --- evaluated at equilibrium prices. Then it must be more profitable (and attainable) to the firm sector as well. Then it must be available and more profitable to some firm. But that contradicts the definition of CE.

Proof: $u^i(w^{0i}) \geq u^i(x)$, for all x so that $p^0 \bullet x \leq M^i(p^0)$, for all $i \in H$. By local non-satiation (follows from C.IV and C.VI(C)), $p^0 \bullet x = M^i(p^0)$. Then

- If $u^i(x) > u^i(w^{0i})$, for typical,

$i \in H$, then $p^0 \cdot x > p^0 \cdot w^{0i}$.

- $p^0 \cdot y > p^0 \cdot y^{oj}$ implies $y \notin Y^j$.
- $\sum_{i \in H} w^{0i} \leq \sum_{j \in F} y^{0j} + r$.
- For each $i \in H$, $p^0 \bullet w^{0i} = M^i(p^0) = p^0 \cdot r^i + \sum_j \alpha^{ij}(p^0 \cdot y^{0j})$, (by C.IV

combined with C.VI(C) or just C.IV*) and summing over households,

$$\sum_{i \in H} p^0 \bullet w^{0i} = \sum_i M^i(p^0) = \sum_i \left[p^0 \cdot r^i + \sum_j \alpha^{ij}(p^0 \cdot y^{oj}) \right]$$

$$= p^0 \cdot \sum_i r^i + p^0 \cdot \sum_i \sum_j \alpha^{ij} y^{0j}$$

$$= p^0 \cdot \sum_i r^i + p^0 \cdot \sum_j \sum_i \alpha^{ij} y^{0j}$$

$$= p^0 \cdot r + p^0 \cdot \sum_j y^{0j} \quad \text{(since for each j, } \sum_i \alpha^{ij} = 1 \text{).}$$

Proof by contradiction. Suppose, contrary to the theorem, there is an attainable allocation v^i, $i \in H$, so that $u^i(v^i) \geq u^i(w^{0i})$ all i with $u^h(v^h) > u^h(w^{0h})$ for some

$h \in H$. The allocation v^i must be more expensive than w^{0i} for those households made better off and no less expensive for the others. Then we have

$$\sum_{i\in H} p^0 \cdot v^i > \sum_{i\in H} p^0 \cdot w^{0i} = \sum_{i\in H} M^i(p^0) = p^0 \cdot r + p^0 \cdot \sum_{j\in F} y^{0j} \quad .$$

But if v^i is attainable, then there is $y'^j \in Y^j$ for each $j \in F$, so that

$$\sum_{i\in H} v^i \leq \sum_{j\in F} y'^j + r$$

, (co-ordinatewise). But then, evaluating this production plan at the equilibrium prices, p^o, we have

$$p^0 \cdot r + p^0 \cdot \sum_{j\in F} y^{0j} < p^0 \cdot \sum_{i\in H} v^i \quad \leq p^0 \cdot \sum_{j\in F} y'^j + p^0 \cdot r \quad .$$

So $p^0 \cdot \sum_{j\in F} y^{0j} < p^0 \cdot \sum_{j\in F} y'^j$. Therefore for some $j \in F$, $p^0 \cdot y^{0j} < p^0 \cdot y'^j$.

But y^{0j} maximizes $p^0 \cdot y$ for all $y \in Y^j$; there cannot be $y'^j \in Y^j$ so that $p \cdot y'^j > p \cdot y^{0j}$. This is a contradiction. Hence, $y'^j \notin Y^j$. The contradiction shows that v^i is not attainable. Q.E.D.

1FTWE does not require convexity.

Second Fundamental Theorem of Welfare Economics (2FTWE)

(Every PE can be supported as CE subject to income redistribution. Requires convexity). We prove this in two steps, first that there are supporting prices (Thm. 19.2), and second that there is a way to parse endowment and ownership to make budgets balance (this is just bookkeeping, Corollary 19.1). Note that convexity of possible consumption sets (C.III), X^i, of preferences (C.VI(C)), and of technology (P.I) are essential to this result.

Recall: **Theorem 8.2 (Separating Hyperplane Theorem)**: Let A, B $\subset R^N$; let A and B be nonempty, convex, and disjoint, that is $A \cap B = \phi$. Then there is $p \in R^N$, $p \neq 0$, so that $p \cdot x \geq p \cdot y$, for all $x \in A$, $y \in B$.

Let $A^i(x^i) \equiv \{x | x \in X^i, \quad u^i(x) \geq u^i(x^i)\}$.

Theorem 19.2: Assume P.I-P.IV and C.I-C.V, C.VI(C). Let x^{*i}, y^{*j} , $i \in H$, $j \in F$, be an attainable Pareto efficient allocation. Then there is $p \in P$ so that

(i) x^{*i} minimizes $p \cdot x$ on $A^i(x^{*i})$, $i \in H$, and

(ii) y^{*j} maximizes $p \cdot y$ on Y^j , $j \in F$.

Proof: Let $x^* = \sum_{i \in H} x^{*i}$, and let $y^* = \sum_{j \in F} y^{*j}$. Note that $x^* \leq y^* + r$ (the inequality applies co-ordinatewise). Let $A = \sum_{i \in H} A^i(x^{*i})$. Let $B = \sum_{j \in F} Y^j + \{r\} = Y + \{r\}$. A and B are closed convex sets with common points, x^*, $y^* + r$.

Let $\mathbf{A} = \sum_{i \in H} \{x \mid x \in X^i, u^i(x) > u^i(x^{*i})\}$. A = closure ($\mathbf{A}$).

$\mathbf{A}$ and B are disjoint, convex. By the Separating Hyperplane Theorem, there is a normal p , so that $p \cdot x \geq p \cdot v$ for all $x \in \mathbf{A}$, and all $v \in B$. By continuity of u^i, all i, and continuity of the dot product we have also $p \cdot x \geq p \cdot v$ for all $x \in A$ and all $v \in B$ so that $p \cdot x^* \geq p \cdot (y^* + r)$. $p \geq 0$, by (C.IV), and $x^* \leq y^* + r$, so $p \cdot x^* \leq p \cdot (y^* + r)$.

Thus x* and (y^*+r) minimize $p \cdot w$ on A and maximize $p \cdot w$ on B. Without loss of generality, let $p \in P$. Then --- based on the additive structure of A and B, x^{*i} minimizes $p \cdot x$ on $A^i(x^{*i})$and y^{*j} maximizes $p \cdot y$ on Y^j. That is,

$$p \cdot x^* = \min_{x \in A} p \cdot x = \min_{x^i \in A^i(x^{*i})} p \cdot \sum_{i \in H} x^i = \sum_{i \in H} \left(\min_{x \in A^i(x^{*i})} p \cdot x \right) \text{ , and}$$

$$p \cdot (r + y^*) = \max_{v \in B} p \cdot v = p \cdot r + \max_{y^j \in Y^j,\, j \in F} p \cdot \Sigma\, y^j = p \cdot r + \sum_{j \in F} \left(\max_{y^j \in Y^j} p \cdot y^j \right) .$$

So x^{*i} minimizes $p \cdot x$ for all $x \in A_i(x^{*i})$ and y^{*j} maximizes $p \cdot y$ for all $y \in Y^j$. QED

Corollary 19.1 (Second Fundamental Theorem of Welfare Economics): Assume P.I-P.IV, and C.I-C.VI(C). Let x^{*i} , y^{*j} be an attainable Pareto efficient allocation. Then there is $p \in P$ and a choice $\hat{r}^i \geq 0,\ \hat{\alpha}^{ij} \geq 0$ so that

$$\sum_{i \in H} \hat{r}^i = r$$

$$\sum_{i \in H} \hat{\alpha}^{ij} = 1 \text{ for each j, and}$$

$$p \cdot y^{*j} \text{ maximizes } p \cdot y \text{ for } y \in Y^j$$

$$p \cdot x^{*i} = p \cdot \hat{r}^i + \sum_{j \in F} \alpha^{ij}(p \cdot y^{*j})$$

and (Case 1, $p \cdot x^{*i} > \min_{x \in X^i} p \cdot x$) $u^i(x^{*i}) \geq u^i(x)$ for all $x \in X^i$ so that $p \cdot x \leq p \cdot \hat{r}^i + \sum_{j \in F} \hat{\alpha}^{ij}(p \cdot y^{*j})$

or (Case 2, $p \cdot x^{*i} = \min_{x \in X^i} p \cdot x$) x^{*i} minimizes $p \cdot x$ for all x so that $u^i(x) \geq u^i(x^{*i})$.

Proof: By Theorem 19.2, there is $p \in P$ so that y^{*j} maximizes $p \cdot y$ for all $y \in Y^j$, and so that x^{*i} minimizes $p \cdot x$ for all $x \in A^i(x^{*i})$.

By attainability,

$\sum_{i \in H} x^{*i} \leq \sum_{j \in F} y^{*j} + r$. Multiplying through by p, with the recognition of free goods, we have

$$\sum_{i \in H} p \cdot x^{*i} = \sum_{j \in F} p \cdot y^{*j} + p \cdot r$$

Let $\lambda_i = \dfrac{p \cdot x^{*i}}{\sum_{h \in H} p \cdot x^{*h}}$, and set $\hat{r}^i = \lambda_i r$, $\hat{\alpha}^{ij} = \lambda_i$, for all $i \in H$, $j \in F$. Then

$$p \cdot x^{*i} = p \cdot \hat{r}^i + \sum_{j \in F} \hat{\alpha}^{ij} p \cdot y^{*j} \ .$$

Now show that cost minimization subject to utility constraint is equivalent to utility maximization subject to a budget constraint (in case 1). This follows from continuity of u^i. Suppose, on the contrary, there is x'^i so that $p \cdot x'^i = p \cdot x^{*i}$ and $u^i(x'^i) > u^i(x^{*i})$. By continuity of u^i, C.V, there is an ε neighborhood about x'^i so that all points in the neighborhood have higher utility than x^{*i} . But then some points of the neighborhood are less expensive at p than x^{*i} , and x^{*i} is no longer a cost minimizer for $A_i(x^{*i})$. This is a contradiction, hence there can be no such x'^i.

The assertion for case 2 is merely a restatement of the property shown in Theorem 19.2.

QED